REMARKS ON REDUCTIVE OPERATOR ALGEBRAS

BY

ABIE FEINTUCH AND PETER ROSENTHAL

ABSTRACT

It is shown that a weakly closed operator algebra with the property that each of its invariant subspaces is reducing and which is either strictly cyclic or has only closed invariant linear manifolds, must be a von Neumann algebra.

1. Introduction

Let $\mathscr{H}$ be a complex Hilbert space. The well known *transitive algebra problem* is the question: must a weakly closed algebra of operators which contains the identity and whose only invariant subspaces are $\{0\}$ and $\mathscr{H}$ be the algebra of all operators on $\mathscr{H}$? In the case where $\mathscr{H}$ is finite-dimensional, the affirmative answer to the above is Burnside's Theorem (see [2, p. 101]).

Arveson's work [1] on the transitive algebra problem led to research on this question by a number of authors ([3], [7], [9], [10], [11]). Many partial results have been obtained, but the problem is still unsolved in general.

The *reductive algebra problem*, raised in [12], is the question: if $\mathfrak{A}$ is a weakly closed algebra of operators on $\mathscr{H}$ which contains the identity and which has the property that all of its invariant subspaces are reducing, must $\mathfrak{A}$ be self-adjoint? As observed in [12], an affirmative answer to this question would imply an affirmative answer to the transitive algebra problem. Partial results have been obtained in [8], [14] and [12], but the reductive algebra problem is also still unsolved.

Here we present several other results on the reductive algebra problem. We show that a reductive algebra $\mathfrak{A}$ with the property that every densely defined graph transformation of $\mathfrak{A}$ is bounded, must be self-adjoint. This gives generalizations of the results of Foias ([5], [6]), Herrero [7] and Lambert [9] concerning

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transitive algebras, and also yields certain other special cases of the reductive algebra problem.

2. Preliminaries

In the following, an *operator algebra* is a weakly closed algebra of bounded operators on a Hilbert space which contains the identity. If $\mathfrak{A}$ is an operator algebra, then Lat $\mathfrak{A}$ denotes the collection of all closed linear subspaces of the Hilbert space which are invariant under $\mathfrak{A}$. The operator algebra $\mathfrak{A}$ is *reductive* if $\mathcal{M} \in \operatorname{Lat} \mathfrak{A}$ implies $\mathcal{M}^{\perp} \in \operatorname{Lat} \mathfrak{A}$, or, equivalently, if $\operatorname{Lat} \mathfrak{A} = \operatorname{Lat} \mathfrak{A}^*$ (where $\mathfrak{A}^* = \{A^* : A \in \mathfrak{A}\}$). If $\mathcal{H}$ is a Hilbert space and n is a positive integer, then $\mathcal{H}^{(n)}$ denotes the direct sum of n copies of $\mathcal{H}$, and if A is an operator on $\mathcal{H}$, then $A^{(n)}$ denotes the direct sum of n copies of A, acting on $\mathcal{H}^{(n)}$ in the standard fashion.

If $\mathfrak{A}$ is an algebra of operators on $\mathcal{H}$, then $\mathfrak{U}^{(n)} = \{A^{(n)} : A \in \mathfrak{A}\}$. We use $\mathfrak{A}'$ to denote the commutant of $\mathfrak{A}$.

If $\mathfrak{A}$ is an operator algebra and $\mathcal{M} \in \operatorname{Lat} \mathfrak{A}^{(n)}$, then $\mathcal{M}$ is an *invariant graph subspace for* $\mathfrak{A}^{(n)}$ if there exist linear transformations $T_1, \cdots, T_{n-1}$ with a common domain $\mathcal{D}$, ($\mathcal{D}$ a linear manifold different from $\{0\}$ in $\mathcal{H}$) such that

$$\mathcal{M} = \{x \oplus T_1 x \oplus \cdots \oplus T_{n-1} x : x \in \mathcal{D}\}.$$

A linear transformation T is a *graph transformation for* $\mathfrak{A}$ if, for some n, T occurs as one of the T_i's in an invariant graph subspace for $\mathfrak{A}^{(n)}$.

3. The main result

THEOREM. *Let $\mathfrak{A}$ be a reductive algebra on $\mathcal{H}$. If every densely defined graph transformation of $\mathfrak{A}$ is bounded, then $\mathfrak{A}$ is self-adjoint.*

We have divided the proof of the above theorem into a series of lemmas.

The following is a special case of an easy, well-known result.

LEMMA 1. *If $\mathfrak{A}^{(n)}$ is reductive for every positive integer n, then $\mathfrak{A}$ is self-adjoint.*

PROOF. See [12, Lemma 2].

LEMMA 2. *If $\mathfrak{A}$ is reductive, $\mathfrak{A}' = (\mathfrak{A}^*)'$, and every densely-defined graph transformation for $\mathfrak{A}$ is bounded, then $\mathfrak{A}$ is self-adjoint.*

PROOF. The basic ideas of the proof are from [12]. By Lemma 1, it suffices to show that $\mathcal{M} \in \operatorname{Lat} \mathfrak{A}^{(n)}$ implies $\mathcal{M} \in \operatorname{Lat} (\mathfrak{A}^*)^{(n)}$ for each positive integer n. This holds for $n = 1$ by hypothesis; we proceed by induction. Assume that the case n is known and let $\mathcal{M} \in \mathrm{Lat}\, \mathfrak{A}^{(n+1)}$. If $\mathcal{N}$ is the subspace of $\mathcal{M}$ consisting of all vectors whose first component is 0, then $\mathcal{N} \in \mathrm{Lat}\, (\mathfrak{A}^{(n+1)} \cap \mathrm{Lat}\, (\mathfrak{A}^*)^{(n+1)}$ by the induction hypothesis. Now $\mathcal{M} \ominus \mathcal{N} \in \mathrm{Lat}\, (\mathfrak{A}^{(n+1)}$, and $\mathcal{M} \ominus \mathcal{N}$ is a graph subspace (see [12]);

$$\mathcal{M} \ominus \mathcal{N} = \{x \oplus T_1 x \oplus \cdots \oplus T_n x : x \in \mathcal{D}\}$$

for some linear manifold $\mathcal{D}$ and linear transformations $\{T_i\}$ with domain $\mathcal{D}$. Then $\mathcal{D}$ is invariant under $\mathfrak{A}$ and $AT_i = T_i A$ for all $A \in \mathfrak{A}$.

Let P denote the projection onto $\mathcal{D}'$ and define $\tilde{T}_i$ by $\tilde{T}_i x = T_i P x$ for all x such that $Px \in \mathcal{D}$. Then

$$\{x \oplus \tilde{T}_1 x \oplus \cdots \oplus \tilde{T}_n x : Px \in \mathcal{D}\}$$
$$= \mathcal{M} \ominus \mathcal{N} \oplus \{y \oplus 0 \oplus \cdots \oplus 0 : y \in \mathcal{D}^\perp\}$$

is an invariant subspace of $\mathfrak{A}^{(n-1)}$. Hence $\tilde{T}_i$ is a densely defined graph transformation for $\mathfrak{A}$, and $\tilde{T}_i$ is bounded for each i. It follows that each $\tilde{T}_i$ is in $\mathfrak{A}' = (\mathfrak{A}^*)'$, and, since $P \in \mathfrak{A}'$, this implies that $T_i A^* = A^* T_i$ for each $A \in \mathfrak{A}$. Since $\mathcal{M} \ominus \mathcal{N}$ is a closed subspace, $\mathcal{D}$ is closed and reduces $\mathfrak{A}$. This implies that $\mathcal{M} \ominus \mathcal{N} \in \mathrm{Lat}(\mathfrak{A}^*)^{(n+1)}$. Thus $\mathcal{M} = (\mathcal{M} \ominus \mathcal{N}) \oplus \mathcal{N}$ is in $\mathrm{Lat}\, (\mathfrak{A}^*)^{(n+1)}$.

To prove the Theorem we need only to show that the hypothesis $\mathfrak{A}' = (\mathfrak{A}^*)'$ in Lemma 2 is superfluous.

Lemma 3. *Let $\mathfrak{A}$ be a reductive algebra. If $T \in \mathfrak{A}'$ and $T^2 = 0$, then $T \in (\mathfrak{A}^*)'$.*

Proof. Suppose T satisfies the hypothesis, and let $\mathcal{K}$ denote the null space of T. Then T has the form

$$\begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}$$

with respect to the decomposition $\mathcal{K} \oplus \mathcal{K}^\perp$ of $\mathcal{H}$. Now $\mathcal{K}$ is invariant, and hence reducing, for $\mathfrak{A}$, and therefore $S \in \mathfrak{A}$ implies

$$S = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}$$

with respect to this decomposition. Since

$$\begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix} \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix} = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix} \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}$$

it follows that $CS_2 = S_1C$. Thus the subspace $\{Cx \oplus x: x \in \mathscr{K}^{\perp}\}$ is in Lat $\mathfrak{A}$ and is invariant under $\mathfrak{A}^*$. If

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

this shows that $CA_2^* = A_1^*C$, and therefore $T \in (\mathfrak{A}^*)'$.

The following lemma is implicitly contained in [6], [7], and [9].

LEMMA 4. *Let $\mathfrak{A}$ be an operator algebra with the property that every closed densely defined linear transformation that commutes with $\mathfrak{A}$ is bounded. If $T \in (\mathfrak{A})'$, then*

$$\sigma(T) = \Pi_0(T) \cup \overline{\Pi_0(T^*)},$$

(where Π_0 denotes point spectrum and "$-$" denotes complex conjugation).

PROOF. Suppose $T \in (\mathfrak{A})'$ and $\lambda \in \sigma(T)$. If $\lambda \notin \Pi_0(T) \cup \Pi_0(T^*)$, then $(T - \lambda)$ is one to one and has dense range. Then $(T - \lambda)^{-1}$ is a well-defined closed linear transformation with dense domain, and clearly $(T - \lambda)^{-1}$ commutes with $\mathfrak{A}$. The hypothesis implies that $(T - \lambda)^{-1}$ is bounded, which contradicts the fact that $\lambda \in \sigma(T)$.

LEMMA 5. *If $\mathfrak{A}$ is reductive and every closed densely defined linear transformation that commutes with $\mathfrak{A}$ is bounded, then every collection of mutually orthogonal nontrivial members of* Lat $\mathfrak{A}$ *is finite.*

PROOF. If this were not the case, then there would exist infinitely many mutually orthogonal nontrivial reducing subspaces $\{\mathscr{N}_j\}_{j=1}^{\infty}$ for $\mathfrak{A}$. Let $\mathscr{N}_0 = \bigcap_{j=1}^{\infty}\{\mathscr{N}_j^{\perp}\}$ and $\mathscr{M}$ be the set of all vectors $x = \sum_{j=0}^{\infty} \oplus x_j$ with $x_j \in \mathscr{N}_j$ and

$$\sum_{j=1}^{\infty} |j|^2 \|x_j\|^2 < \infty.$$

Then $\mathscr{M}$ is obviously invariant under $\mathfrak{A}$. Moreover, $\mathscr{M}$ is the domain of the closed operator

$$T = \sum_{j=1}^{\infty} \oplus\, jI_j \text{ on } \sum_{j=0}^{\infty} \oplus\, \mathscr{N}_j,$$

where I_j is the identity operator on $\mathscr{N}_j$. Since $\mathscr{M}$ is dense in $\mathscr{H}$ and T commutes with $\mathfrak{A}$, T is bounded by hypothesis, which is clearly a contradiction.

PROOF OF THEOREM. By Lemma 2, it suffices to show that $T \in \mathfrak{A}'$ implies $T \in (\mathfrak{A}^*)'$. So suppose $T \in \mathfrak{A}'$ and let $\lambda_1 \in \sigma(T)$. Then λ_1 is an eigenvalue of T or

$\bar{\lambda}_1$ is an eigenvalue of T^*. If E_{λ_1} is the eigenspace (of either T or T^*) corresponding to λ_1, then E_{λ_1} reduces $\mathfrak{A}$ and $\mathfrak{A}^*$. If P_1 is the projection of $\mathscr{H}$ onto E_{λ_1}, P_1TP_1 is a multiple of P_1.

Now consider the compression T_1 of T to $(E_{\lambda_1})^{\perp}$. Since $\mathfrak{A}|E_{\lambda_1}^{\perp}$ satisfies the hypothesis of Lemma 4, (Lemma 4 does not require $\mathfrak{A}$ closed), we can find $\lambda_2 \in \sigma(T_1)$ such that λ_2 or $\bar{\lambda}_2$ is in $\Pi_0(T_1)$. If E_{λ_2} is the eigenspace corresponding to λ_2 or $\bar{\lambda}_2$ and P_2 is the projection of $\mathscr{H}$ onto E_{λ_2}, then E_{λ_2} reduces $\mathfrak{A}$ and P_2TP_2 is a multiple of P_2. Now consider the compression of T to $(E_{\lambda_1} \oplus E_{\lambda_2})^{\perp}$, and produce an E_{λ_3} as above.

It follows from Lemma 5 that the procedure indicated above will terminate after a finite number of steps and produce a set $\{E_{\lambda_i}\}_{i=1}^{n}$ of mutually orthogonal subspaces such that $\sum_{i=1}^{n} \oplus E_{\lambda_i} = \mathscr{H}$ and for each i, P_iTP_i is a scalar multiple of P_i.

Now $T = \sum_{i,j=1}^{n} P_iTP_j$ and since, for each i, $P_i \in \mathfrak{A}' \cap (\mathfrak{A}^*)'$, it is enough to show that $P_iTP_j \in (\mathfrak{A}^*)'$ for $i \neq j$. But in this case $(P_iTP_j)^2 = 0$ and Lemma 3 applies. This completes the proof.

4. Corollaries

As mentioned in the introduction, a solution to the reductive algebra problem leads to a solution to the transitive algebra problem. Our first corollary is essentially known, (cf. [6], [9]).

COROLLARY 1. *Let $\mathfrak{A}$ be a transitive algebra on $\mathscr{H}$. If every densely defined graph transformation of $\mathfrak{A}$ is bounded, then $\mathfrak{A} = B(\mathscr{H})$.*

It is well known (see, for example, [13, p. 61]) that strictly transitive algebras are strictly dense in $B(\mathscr{H})$. The next corollary is essentially a generalization of this fact. It includes, in particular, the (already-known) solution of the reductive algebra problem in the finite-dimensional case (cf. [2], [12]).

COROLLARY 2. *If $\mathfrak{A}$ is reductive and every invariant linear manifold of $\mathfrak{A}$ is closed, then $\mathfrak{A}$ is self-adjoint.*

PROOF. Let T be a densely defined graph transformation of $\mathfrak{A}$ with domain $\mathscr{D}$. Then, since $\mathscr{D}$ is invariant under $\mathfrak{A}$, $\mathscr{D}$ is closed. If $\mathscr{M} = \{x \oplus Tx \oplus \cdots \oplus T_nx: x \in \mathscr{D}\}$ is the corresponding member of Lat $\mathfrak{A}^{(n+1)}$, then the map $x \to Tx \oplus T_2x \oplus \cdots \oplus T_nx$ is bounded by the closed graph theorem, and thus T is bounded.

A linear manifold $\mathscr{L}$ of $\mathscr{H}$ is an *operator range* if there exists a Hilbert space $\mathscr{K}$

and a bounded operator A from $\mathcal{K}$ into $\mathcal{H}$ such that $\mathcal{L} = A\mathcal{K}$. The next corollary generalizes the result of Foias ([5], [6]) concerning transitive algebras. An excellent account of the known facts about operator ranges is given in [4].

COROLLARY. 3. *Let $\mathfrak{A}$ be a reductive algebra such that every invariant operator range of $\mathfrak{A}$ is closed. Then $\mathfrak{A}$ is self-adjoint.*

PROOF. If $\mathcal{D}$ is the domain of any graph transformation T of $\mathfrak{A}$, then $\mathcal{D}$ is an operator range and therefore is closed. Thus, as in Corollary 2, T is bounded.

An operator algebra $\mathfrak{A}$ is *of finite strict multiplicity* if there exists a finite set $\{x_i\}_{i=1}^n$ of vectors such that the linear span of $\{Ax_i: A \in \mathfrak{A}\}$ is $\mathcal{H}$.

Herrero [7] has generalized the result of Lambert [9] in the case $n = 1$ to show that a transitive algebra containing an algebra $\mathfrak{A}$ of finite strict multiplicity is $B(\mathcal{H})$.

COROLLARY 4. *A reductive algebra containing an algebra of finite strict multiplicity is self-adjoint.*

PROOF. By a result of Herrero [7, Lemma 1], every graph transformation of $\mathfrak{A}$ is bounded, and the theorem applies.

REMARK. (i) Lemma 5 and an argument similar to that in [12, Th. 1] show that a von Neumann algebra with the property that every closed densely defined linear transformation commuting with it is bounded must be a direct sum of a finite number of type I factors.

(ii) It follows easily from Lemma 3 that if A is an algebraic operator which commutes with a reductive algebra $\mathfrak{A}$ then $A \in (\mathfrak{A}^*)'$. For such an A can be written in the form

$$A = \begin{bmatrix} \lambda_1 & A_{12} & \cdots & A_{1n} \\ & \lambda_2 & \ddots & \\ & 0 & \ddots & A_{n-1\,n} \\ & & & \lambda_n \end{bmatrix}$$

with respect to a decomposition of $\mathcal{H}$ into a direct sum $\sum_{i=1}^n \oplus \mathcal{H}_i$ where each $\mathcal{H}_i$ reduces $\mathfrak{A}$. Then if P_i is the projection of $\mathcal{H}$ onto $\mathcal{H}_i$, $P_iA\,P_j \in \mathfrak{A}'$, and by Lemma 3, $P_iAP_j \in (\mathfrak{A}^*)'$. Hence $A = \sum P_iAP_j$ is in $(\mathfrak{A}^*)'$.

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UNIVERSITY OF TORONTO
TORONTO, CANADA
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BE'ER SHEVA, ISRAEL